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Dominating projective sets in the Baire space

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Abstract

We show that every analytic set in the Baire space which is dominating contains the branches of a *uniform tree*, i.e. a superperfect tree with the property that for every splitnode all the successor splitnodes have the same length. We call this property of analytic sets *u-regularity*. However, we show that the concept of uniform tree does not suffice to characterize dominating analytic sets in general. We construct a dominating closed set with the property that for no uniform tree whose branches are contained in the closed set, the set of these branches is dominating. We also show that from a Σ^1_{n+1}-rapid filter a non-u-regular Π^1_n-set can be constructed. Finally, we prove that Σ^1_2-K_σ-regularity implies Σ^1_2-u-regularity.

0. Introduction

The Cantor–Bendixson theorem says that a closed set in a Polish space either is countable or contains a perfect subset. Suslin (see [9]) proved that this is true even for analytic sets.

In this paper we will study a similar regularity property for analytic sets in the Baire space ${}^\omega\omega$, i.e. the set of all functions from ω to ω endowed with the product topology of the discrete topology on ω. It is well known that a closed set in the Baire space is the set of all infinite branches of a tree on ${}^{<\omega}\omega$ (the set of all finite sequences of integers) and conversely.

On the Baire space the ordering $\leqslant^*$ is defined by $x \leqslant^* y$ if and only if $x(n) \leqslant y(n)$ for all but finitely many n's. A subset $A \subseteq {}^\omega\omega$ is called *bounded/unbounded* if and only if A has an/no upper bound in $\langle {}^\omega\omega, \leqslant^* \rangle$, respectively. A is called *cofinal* if and only if for every $x \in {}^\omega\omega$, there exists $y \in A$ such that $y \geqslant^* x$. Cofinal sets in $\langle {}^\omega\omega, \leqslant^* \rangle$ are usually called *dominating*.

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In [8], Kechris proved that every analytic set in the Baire space is either bounded or contains the branches of a superperfect tree on ${}^{<\omega}\omega$, i.e. an instance of a special type of closed set. This property of analytic sets is called *K_σ-regularity*. A *superperfect tree* on ${}^{<\omega}\omega$ is a tree with the property that every node has an extension which splits into infinitely many successor nodes.

It is natural to ask whether there exists a "thicker" type of closed set than that of superperfect tree such that every analytic set either is not dominating or contains such a closed set. A set theorist might think that the notion of Laver tree might be the right one for this. However, it is not difficult to see that this is not the case. We will prove the following theorem:

Theorem 1. *Let $A \subseteq {}^{<\omega}\omega$ be an analytic set. Then either A is not dominating or A contains all the branches of a uniform tree.*

Here a *uniform tree* is a superperfect tree with the property that for each splitnode all its successor splitnodes have the same length. For closed sets this result is implicit in [3]. In [3] abstract graphs have been investigated, and it is asked when a graph is dominating in the sense that its vertices can be labelled with integers in such a way that the labellings along its rays form a dominating family of functions in ${}^{\omega}\omega$. It is shown that a graph is dominating if and only if it contains a certain number of disjoint copies of one out of three special types of graph. The first type is a ray, the second type corresponds to a superperfect tree, and the third type corresponds to what is called a uniform tree here. Whereas the graph corresponding to a uniform tree is dominating in the sense of [3], in general the set of all branches of a uniform tree is not dominating in the usual sense.

The main method of proof in [3] is to try to define a rank function on all the vertices of a graph such that – if one vertex does not get a rank a subgraph of the appropriate type can be constructed, and – if all vertices get ranked, then for every labelling, from the rank function a function in ${}^{\omega}\omega$ can be constructed which is not dominated by any ray. This method yields Theorem 1 for closed A but does not generalize.

As in [8], our method of proof will be to define an appropriate game for $A \subseteq {}^{\omega}\omega$ such that from a winning strategy for player I a uniform tree can be constructed through A, and from a winning strategy for player II a function in ${}^{\omega}\omega$ can be derived which is not dominated by any member of A. From Martin's famous result [10] that Borel games are determined we obtain Theorem 1 for Borel sets. Using a variant of Solovay's "unfolding trick" we show that it even holds for analytic sets.

Since clearly the branches of a superperfect tree are an unbounded family, Kechris' result gives a topological characterization of unbounded analytic sets. However, the branches of a uniform tree form a dominating family only in special cases, e.g. if all nodes are non-decreasing. Hence, Theorem 1 gives a topological characterization of dominating analytic sets in the G_δ-subspace of the Baire space consisting of all non-decreasing functions:

Corollary 2. *An analytic set in the Baire space containing only nondecreasing functions is dominating if and only if it contains the branches of a uniform tree.*

One might expect that a specialization of the concept of uniform tree to one of dominating uniform tree should suffice to give a characterization in the general case. Surprisingly this is not true.

Theorem 3. *There exists a dominating closed set $A \subseteq {}^{\omega}\omega$ such that for every uniform tree it is true that if X is the set of all its branches and $X \subseteq A$, then X is not dominating.*

A set $A \subseteq {}^{\omega}\omega$ is called K_σ-regular if it is either bounded or contains the branches of a superperfect tree. Analogously, we call A *u-regular*, if either it is not dominating or it contains the branches of a uniform tree. So Theorem 1 says that every analytic set is *u-regular.*

Since the proofs of Kechris' theorem and of Theorem 1 use a game argument (projective) determinacy implies K_σ- as well as u-regularity of all (projective) sets. However, in [7] there is an example of a dominating set in Gödel's universe L which is Π^1_1 in V and does not contain a perfect subset. Using ideas from [6] we prove the following theorem.

Theorem 4. *If there exists a Σ^1_{n+1}-rapid filter then there exists a non-u-regular Π^1_n-set.*

As Σ^1_2-rapid filters exist in L we obtain another example of a non-u-regular Π^1_1-set in L. However, as it might contain a perfect set it is not as good as Kechris'.

It has become popular to compare the strength of Σ^1_2-regularity for different regularity properties. For a property P we say Σ^1_2-P-regularity holds if every Σ^1_2-set has P. In [1] and [12] it is proved that Σ^1_2-measurability implies Σ^1_2-Baire property. Judah (in [5]) proved that Σ^1_2-Baire property as well as Σ^1_2-Ramsey property imply Σ^1_2-K_σ-regularity. We will prove the following theorem.

Theorem 5. *Σ^1_2-K_σ-regularity implies Σ^1_2-u-regularity.*

We do not know whether Σ^1_2-u-regularity is equivalent to K_σ-regularity.

1. Every dominating analytic set contains the branches of a uniform tree

We start with fixing our notation. We will consider only trees $p \subseteq {}^{<\omega}\omega$ which have a stem, denoted by stem (p), i.e. a maximal node $s \in p$ such that $\forall t \in p (t \subseteq s \ \vee \ s \subseteq t)$. A node $s \in p$ is called a split-node if and only if $\{n \in \omega : s^\wedge \langle n \rangle \in p\}$ has more than one element. The set of all splitnodes of p is denoted by Split(p). For $s \in$ Split(p) we denote by $\mathrm{Succ}_p(s)$ the set of all successor splitnodes of s in p, i.e. the set of all $t \in$ Split(p) such that $s \subset t$ and for no $s \subset u \subset t$ we have $u \in$ Split(p). By $[p]$ we denote the set of all the maximal branches of the tree p. We say that $s \in {}^{<\omega}\omega$ is *non-decreasing* if and only if $\forall i < j < |s| (s(i) \leqslant s(j))$.

Definition 1.1. A tree $p \subseteq {}^{<\omega}\omega$ is called *superperfect* if and only if for every $s \in p$ there exists $t \in p$ such that $s \subseteq t$ and $\{n\colon t^\wedge\langle n\rangle \in p\}$ is infinite.

Definition 1.2. A tree $p \subseteq {}^{<\omega}\omega$ is called *uniform* if and only if p is superperfect and for every $s \in \mathrm{Split}(p)$, $\{n\colon t^\wedge\langle n\rangle \in p\}$ is infinite and there exists $n \in \omega$ such that for every $t \in \mathrm{Succ}_p(s)$, $|t| = n$. This n (which depends on s) will be denoted by n_s in the sequel.

Uniform trees are related to dominating sets by the following fact:

Fact 1.3. *Let p be a uniform tree containing only nondecreasing nodes. Then $[p]$ is dominating.*

Proof. Let $x \in {}^{\omega}\omega$ be arbitrary. We will find $y \in [p]$ such that for every $n \geqslant \mathrm{stem}(p)$ we have $y(n) > x(n)$.

Let $n = \max\{x(i)\colon |\mathrm{stem}(p)| \leqslant i < n_{\mathrm{stem}(p)}\}$. Since $\mathrm{stem}(p) \in \mathrm{Split}(p)$ we may find $s \in \mathrm{Succ}_p(\mathrm{stem}(p))$ such that $s(|\mathrm{stem}(p)|) > n$. By the assumption on p we conclude for all i, if $|\mathrm{stem}(p)| \leqslant i < |s|$ then $s(i) > x(i)$. As $s \in \mathrm{Split}(p)$ we can repeat this argument to find $s' \in \mathrm{Succ}_p(s)$ (of length n_s) such that for all i if $|\mathrm{stem}(p)| \leqslant i < |s'|$ then $s'(i) > x(i)$. Proceeding similarly we construct y as desired. $\square$

Clearly the assumption that p contains only non-decreasing nodes cannot be dropped in the previous claim. Hence in the following theorem the claimed implication cannot be reversed.

Theorem 1.4. *Let $A \subseteq {}^{\omega}\omega$ be an analytic set. If A is dominating there exists a uniform tree p such that $[p] \subseteq A$.*

Proof. (a) First we prove the theorem in the case that A is a Borel set. We consider the following game $G_u(A)$, where player one plays pairs $\langle s, n\rangle$ in ${}^{<\omega}\omega \times \omega$ and player two plays natural numbers:

I	II
$\langle s_0, n_0\rangle$	
	k_0
$\langle s_1, n_1\rangle$	
	k_1
$\vdots$	$\vdots$

The rules for $G_u(A)$ are

(1) $\forall i < \omega(\langle s_i, n_i \rangle \in {}^{<\omega}\omega \times \omega \setminus \{0\} \wedge k_i \in \omega)$,
(2) $\forall i < \omega(|s_{i+1}| = n_i \wedge s_{i+1}(0) > k_i)$,
(3) player I wins if and only if he keeps rules (1) and (2) and the infinite concatenation of the s_i's he plays belongs to A:

$$s_0{}^\wedge s_1{}^\wedge s_2{}^\wedge \cdots \in A$$

or if player II breaks rule (1).

It is not difficult to verify that if A is Borel $G_u(A)$ is a Borel game. Hence by [10], $G_u(A)$ is determined.

Claim 1. *Suppose that player I has a winning strategy in $G_u(A)$. Then there exists a uniform tree p such that $[p] \subseteq A$.*

Proof of Claim 1. Let p be the tree whose branches are all the possible outcomes of a play where player I follows his winning strategy, say σ. More precisely, if $\sigma: {}^{<\omega}\omega \to {}^{<\omega}({}^{<\omega}\omega \times \omega \setminus \{0\})$ let p be the tree generated by the following set:

$$\{s_0{}^\wedge s_1{}^\wedge \cdots {}^\wedge s_j: (\exists \langle k_0, \ldots, k_j \rangle \in {}^{<\omega}\omega)(\exists \langle n_0, \ldots, n_j \rangle \in {}^{<\omega}\omega)(\forall i < j)$$
$$\langle s_0, n_0 \rangle = \sigma(\emptyset) \wedge \langle s_{i+1}, n_{i+1} \rangle = \sigma(\langle k_0, \ldots, k_i \rangle)\}.$$

As σ is a winning strategy for player I it is easy to see that p contains a uniform tree and $[p] \subseteq A$. □

Claim 2. *Suppose in $G_u(A)$ player II has a winning strategy. Then A is not dominating.*

Proof of Claim 2. Let σ be a winning strategy for player II. So σ is defined on finite sequences of members of ${}^{<\omega}\omega \times \omega$ and has values in ω. Using σ we will define $S: \omega \to \omega$ such that for no $x \in A$, $x \geqslant^* S$ holds. In order that later the proof will go through smoothly we claim that without loss of generality we may assume that σ has been normalized in the sense that it satisfies the following two conditions:

(4) for every $s \in {}^{<\omega}\omega$ the sequence $\langle \sigma(\langle s, n \rangle): n \in \omega \rangle$ is strictly increasing,
(5) for every $s \in {}^{<\omega}\omega$, for every $\langle \langle s_i, n_i \rangle: i < j \rangle$ such that $s = s_0{}^\wedge \cdots {}^\wedge s_{j-1}$ and $(\forall i < j-1)|s_{i+1}| = n_i \wedge n_{j-1} = n$ hold we have

$$\sigma(\langle s, n_{j-1} \rangle) = \sigma(\langle \langle s_i, n_i \rangle: i < j \rangle).$$

In order to see that we may assume (4) and (5) just note that whenever we increase any values of σ then this modified strategy is still a winning strategy for player II; for every play consistent with it is consistent with σ. But now by induction on $|s|$ we may redefine σ such that (4) and (5) hold, as for given n_{j-1} there exist finitely many decompositions of $\langle s, n_{j-1} \rangle$ as in (5) and every such decomposition uniquely determines $\langle s, n_{j-1} \rangle$.

We next define S. In order to define $S(n)$ for $n \in \omega$ we first define $k_0^n, \dots, k_n^n$. Finally, we set $S(n) = k_n^n$. We define by induction:

$$k_0^n = \sigma(\langle \emptyset, n+1 \rangle),$$

$$k_{i+1}^n = \max\{\sigma(\langle s, n+1-(i+1)\rangle): \ |s| = i+1 \wedge (\forall j < i+1) s(j) \leqslant k_j^n\}.$$

Now suppose that $x \in A$ and $n \in \omega$. We want to find $n' \geqslant n$ such that $S(n') \geqslant x(n')$ and so conclude $x \not\geqslant^* S$.

Since we assume that σ has been normalized we may clearly find $l_0 \geqslant n$ minimal such that for all $j \leqslant n$ we have

$$\sigma(\langle x \restriction j, l_0 + 1 - j\rangle) \geqslant x(j). \tag{6}$$

If (6) even holds for every $j \leqslant l_0$ we conclude $S(l_0) \geqslant x(l_0)$, as then by induction it can be proved that for every $j \leqslant l_0$, $x \restriction j$ was a s considered in defining $k_j^{l_0}$, and hence for $j = l_0$ we conclude $S(l_0) = k_{l_0}^{l_0} \geqslant \sigma(\langle x \restriction l_0, 1\rangle) \geqslant x(l_0)$. Otherwise, there exists a minimal j_0 such that $n < j_0 \leqslant l_0$ and

$$\sigma(\langle x \restriction j_0, l_0 + 1 - j_0\rangle) < x(j_0).$$

Next by normality condition (4) we can choose $l_1 > l_0$ minimal such that

$$\sigma(\langle x \restriction j_0, l_1 + 1 - j_0\rangle) \geqslant x(j_0). \tag{7}$$

Again by normality condition (4) and by the minimal choice of j_0, (7) holds for every $j \leqslant j_0$ instead of j_0. If (7) even holds for every $j \leqslant l_1$ instead of j_0, then as before we conclude $S(l_1) \geqslant x(l_1)$ by the definition of S, and we are done.

Otherwise, we let j_1 be minimal such that $j_0 < j_1 \leqslant l_1$ and

$$\sigma(\langle x \restriction j_1, l_1 + 1 - j_1\rangle) < x(j_1) \tag{8}$$

hold. Continuing similarly, we choose $l_2 > l_1$ minimal such that

$$\sigma(\langle x \restriction j_1, l_2 + 1 - j_1\rangle) \geqslant x(j_1) \tag{9}$$

and ask whether (9) even holds for every $j \leqslant l_2$ instead of j_1. If the answer is yes we are done. If it is no we go on constructing j_2 and l_3 and so on.

We claim that at some stage n the answer to this question (where j_{n-1} replaces j_1 and l_n replaces l_2) must be yes, and hence we conclude $S(l_n) \geqslant x(l_n)$ and we are done. For otherwise we would obtain strictly increasing sequences $\langle j_n \colon n < \omega\rangle$ and $\langle l_n \colon n < \omega\rangle$ such that for every $n < \omega$, l_{n+1} is minimal with $\sigma(\langle x \restriction j_n, \ l_{n+1} + 1 - j_n\rangle) \geqslant x(j_n)$, moreover, $j_{n+1} \leqslant l_{n+1}$ and hence $\sigma(\langle x \restriction j_n, \ j_{n+1} - j_n\rangle) < x(j_n)$. But now by normality condition (5) on σ we conclude that the following moves of player I are consistent with σ:

$$\langle x \restriction j_0, j_1 - j_0\rangle, \langle x \restriction [j_0, j_1), j_2 - j_1\rangle, \langle x \restriction [j_1, j_2), j_3 - j_2\rangle, \dots$$

Clearly, the outcome of this play is x. However, since σ is a winning strategy for player II we conclude $x \notin A$, a contradiction.

This proves the theorem in case A is a Borel set.

(b) In the general case where A is analytic, say $A = \pi''[T]$ – where π is the projection and $T \subseteq {}^{<\omega}\omega \times {}^{<\omega}\omega$ is a tree – we use as [8] Solovay's unfolding trick. We consider the modified game $\tilde{G}_u(A)$ in which player I is allowed to play witnesses along the way which will give a branch witnessing that the outcome of the play is the projection of some point in $[T]$. As in part **(a)** the crucial point will be the definition of the unbounded function S. Since we can maximize only over finitely many things we allow player I to wait with playing witnesses. This makes it easier for player I to win but is still enough to ensure that from a winning strategy for him a uniform tree can be constructed through A.

Formally, the game $\tilde{G}_u(A)$ is defined as follows: Player I plays triples in $(\{-1\} \cup \omega) \times {}^{<\omega}\omega \times \omega$. Player II plays natural numbers:

I	II
$\langle w_0, s_0, n_0 \rangle$	
	k_0
$\langle w_1, s_1, n_1 \rangle$	
	k_1
$\vdots$	$\vdots$

The rules are as follows.

(10) For all $i \in \omega, n_i \neq 0$, $|s_{i+1}| = n_i$ and $s_{i+1}(0) > k_i$.

(11) For all $i \in \omega$, $w_i < i$ (hence $w_0 = -1$) and there exist infinitely many $i \in \omega$ such that $w_i \neq -1$.

(12) Let $x = s_0{}^\frown s_1{}^\frown s_2{}^\frown \cdots$, $y = \langle w_0, w_1, \ldots \rangle$, and let y' be the sequence obtained from y by cancelling all w_i's which equal -1 but keeping the order of all the other w_i's. Then I wins if and only if $\langle y', x \rangle \in [T]$.

It is not difficult to see that $\tilde{G}_u(A)$ is a Borel game (since $[T]$ is closed) and hence by [10] is determined.

Claim 1′. *Suppose player I has a winning strategy in $\tilde{G}_u(A)$. Then there exists a uniform tree p such that $[p] \subseteq A$.*

Proof of Claim 1′: As in the proof of Claim 1 it is not difficult to see that the set of all $x \in {}^{\omega}\omega$ such that there exist $\langle w_i \colon i \in \omega \rangle \in {}^{\omega}(\{-1\} \cup \omega)$, $\langle s_i \colon i \in \omega \rangle \in {}^{\omega}({}^{<\omega}\omega)$, $\langle n_i \colon i \in \omega \rangle$ and $\langle k_i \colon i \in \omega \rangle$ in ${}^{\omega}\omega$ such that $x = s_0{}^\frown s_1{}^\frown \cdots$ and

$$\langle w_0, s_0, n_0 \rangle, k_0, \langle w_1, s_1, n_1 \rangle, k_1, \ldots$$

is a play where player I follows his winning strategy is closed, and if p is the tree with branches this set, then p contains a uniform subtree. □

Claim 2′. *Suppose player II has a winning strategy in $\tilde{G}_u(A)$. Then A is not dominating.*

Proof of Claim 2′. Let σ be a winning strategy for player II. Again we may assume that σ has been normalized in the sense that it satisfies the following two conditions:

(13) For every $\langle w,s,n\rangle \in \{-1\} \times {}^{<\omega}\omega \times \omega$ the sequence $\langle \sigma(\langle w,s,n\rangle)\colon n \in \omega\rangle$ is strictly increasing.

(14) For every $\langle w,s,n\rangle$ as in (1) and $\langle\langle w_i,s_i,n_i\rangle\colon i<j\rangle$ such that $s = s_0{}^\frown \cdots {}^\frown s_{j-1}$, $\forall i<j-1(|s_{i+1}| = n_i)$, $n_{j-1} = n$, and $\forall i<j\,(w_i < i)$ we have $\sigma(\langle w,s,n\rangle) = \sigma(\langle\langle w_i,s_i,n_i\rangle\colon i<j\rangle)$.

We next define $S\colon \omega \to \omega$. In order to define $S(n)$ we first define $k_0^n, \ldots, k_n^n$ by induction and then let $S(n) = \max\{k_n^n, n\}$:

$$k_0 = \sigma(\langle -1, \emptyset, n+1\rangle),$$

$$k_{i+1} = \max\{\sigma(\langle -1, s, n+1-(i+1)\rangle)\colon |s| = i+1 \wedge (\forall j < i+1)s(j) \leqslant k_j\}.$$

We will show that S is not bounded by any member of A. Let $x \in A$ and $n \in \omega$. We may assume that x is not bounded by any constant function, since otherwise $S >^* x$ by definition. Choose z such that $\langle z,x\rangle \in [T]$. Now "stretch" z by filling in -1's to obtain y such that $y(0) = -1$ and for all n we have $y(n) < n$ (but without changing the order of the values of z). This is possible by our assumption on x. So $z = y'$ in the notation from rule (12) for the game $\tilde{G}_u(A)$.

Now similarly as in the proof of Claim 2 we find – for given $n \in \omega$ – $n < j_0 < j_1 < \cdots$ and $n < l_0 < l_1 < \cdots$ such that, if for no i $S(l_i) \geqslant x(l_i)$ holds, then

$$\langle y(0), x \restriction j_0, j_1 - j_0\rangle, \langle y(1), x \restriction [j_0,j_1), j_2 - j_1\rangle, \langle y(2), x \restriction [j_1,j_2), j_3 - j_2\rangle, \ldots$$

are moves of player I which are according to the rules (by the choice of y) and consistent with σ (by construction and normality of σ). But clearly this play is lost by player II, a contradiction. □

This completes the proof of Theorem 1.4. □

Definition 1.5. A set $A \subseteq {}^\omega\omega$ is called *K_σ-regular* if and only if either A is $\leqslant^*$-bounded or there exists a superperfect tree p such that $[p] \subseteq A$.

In view of Theorem 1.4 it is natural to introduce a new regularity property.

Definition 1.6. We say that a set $A \subseteq {}^\omega\omega$ is *u-regular* if and only if either A is not dominating or there exists a uniform tree p such that $[p] \subseteq A$.

Corollary 1.7. *Every analytic set is u-regular.*

Since our proof of Theorem 1.4 involves a game argument we clearly have the following corollary.

Corollary 1.8. *The axiom of (projective) determinacy implies that every (projective) set is u-regular.*

By Claim 1.3 and Theorem 1.4 we obtain a topological characterization for dominating analytic subset of the G_δ-subspace of the Baire space consisting of the nondecreasing functions:

Corollary 1.9. *Suppose that $A \subseteq {}^\omega\omega$ is analytic and contains only nondecreasing functions. Then A is dominating if and only if there exists a uniform tree p such that $[p] \subseteq A$.*

2. There exists a dominating tree without any dominating uniform subtree

It is natural to ask whether Theorem 1.4 can be improved to a theorem saying that an analytic set is dominating if and only if it contains the branches of a certain tree. By Corollary 1.9, uniform trees solve this problem in the case that the analytic set contains only nondecreasing functions. One might expect that by specializing the concept of uniform tree to one of dominating uniform tree in an appropriate way the general case should be solved. Surprisingly this is not possible. We have the following theorem:

Theorem 2.1. *There exists a tree $p \subseteq {}^{<\omega}\omega$ such that $[p]$ is dominating but for no uniform subtree $q \subseteq p$, $[q]$ is dominating.*

Proof. The idea is to glue together many nondominating uniform trees in such a way that a dominating tree arises and we keep control over all its uniform subtrees.

For this purpose first we fix the following objects which trivially exist:

Let $\langle k_s : s \in {}^{<\omega}\omega \rangle$ be a family of natural numbers such that

$$\forall s, t \in {}^{<\omega}\omega (k_s \geqslant |s| + 2 \wedge (s \neq t \Rightarrow k_s \neq k_t)).$$

Moreover, let $\langle A_s : s \in {}^{<\omega}\omega \rangle$ be a family of disjoint infinite subsets of ω such that

$$\forall s \forall m \in A_s \ (k_s \leqslant m).$$

Finally, let $\langle l_n : n \in \omega \rangle \in {}^\omega\omega$ be strictly increasing.

Now choose for every $s \in {}^{<\omega}\omega$ a family $\langle P_n^s : n \in A_s \rangle$ such that the following requirements are satisfied (Fig. 1):

(1) $P_n^s \subseteq {}^n\omega \wedge \forall t \in P_n^s (s \subseteq t)$.

(2) $\forall n, m \in A_s \forall t \in P_n^s \forall u \in P_m^s \ (t \neq u \Rightarrow t(|s|) \neq u(|s|))$.

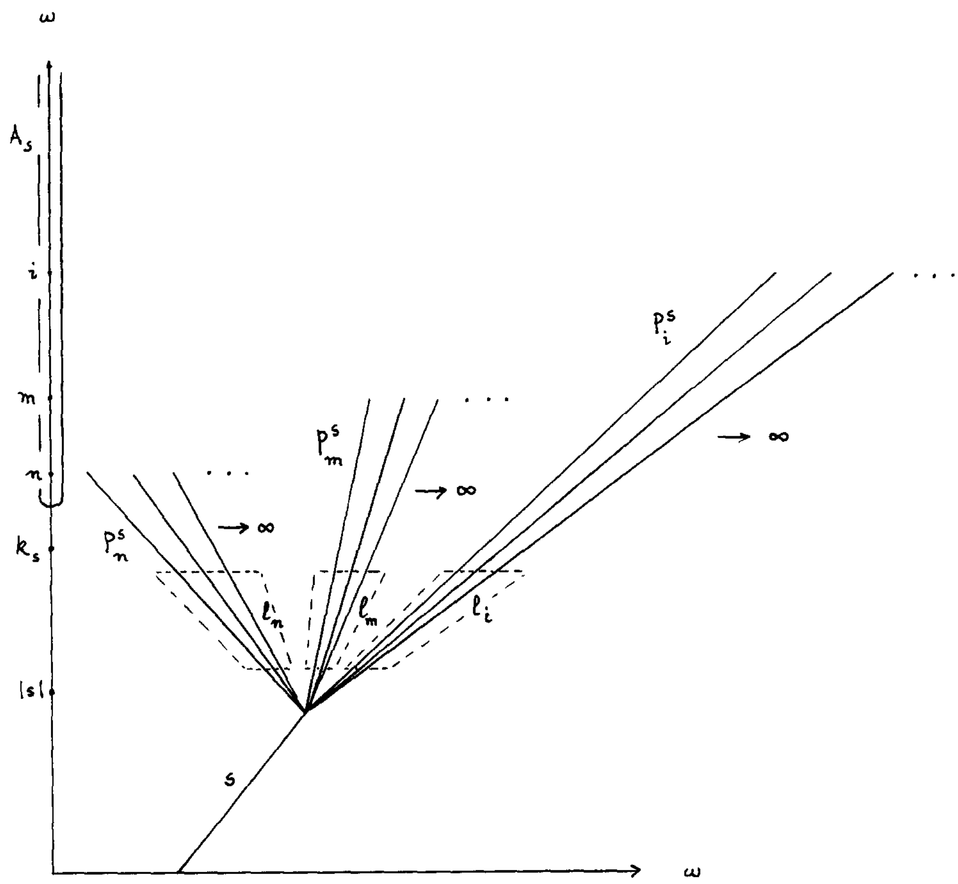

Fig. 1.

(3) $\forall |s|+1 \leqslant i < k_s \forall n \in A_s \forall t \in P^s_n(t(i) = l_n)$.
(4) $\forall n \in A_s \forall m \in \omega \exists t \in P^s_n \forall k_s \leqslant i < n(t(i) > m)$.

Now let p be the unique tree such that $\emptyset \in \mathrm{Split}(p)$ and for every $s \in \mathrm{Split}(p)$

$$\mathrm{Succ}_p(s) = \bigcup \{P^s_n\colon n \in A_s\}.$$

Claim 1. *$[p]$ is dominating.*

Proof of Claim 1. Let $x \in {}^\omega\omega$. Without loss of generality, we may assume that x is increasing. We will find $y \in [p]$ such that $x < y$. We define y by induction as follows:

Choose $n \in A_\emptyset$ so large that $l_n > x(k_\emptyset - 1)$. By (2) and (4) we may find $s_0 \in P^\emptyset_n$ such that $s_0(0) > x(0)$ and $\forall k_\emptyset \leqslant i < n(s_0(i) > x(i))$. Hence, by (3) and our assumption on x we conclude

$$\forall i < n(s_0(i) > x(i)).$$

Suppose now that $s_0 \subset s_1 \subset \cdots \subset s_m$ have been defined such that the following two requirements are satisfied:

(5) $\forall i \leqslant m \exists n \in A_{s_i}(s_{i+1} \in P_n^{s_i})$,
(6) $\forall i \leqslant m \forall j \in \operatorname{dom}(s_i)(s_i(j) > x(j))$.

Find $n \in A_{s_m}$ such that $l_n > x(k_{s_m} - 1)$. By (2) and (4) we may find $s_{m+1} \in P_n^{s_m}$ such that $s_{m+1}(|s_m|) > x(|s_m|)$ and $\forall k_{s_m} \leqslant i < n(s_{m+1}(i) > x(i))$. Hence, by (3), the inductive assumption and our assumption on x we conclude

$$\forall i \in \operatorname{dom}(s_{m+1})(s_{m+1}(i) > x(i)).$$

Finally, we let $y = \bigcup \{s_n : n \in \omega\}$. Clearly, we have $y \in [p]$ and $y > x$. $\square$

Claim 2. *Suppose that $q \subseteq p$ is a uniform subtree. Then $[q]$ is not dominating.*

Proof of Claim 2. Since the sets A_s are chosen pairwise disjoint, by construction it is easy to see that for any $s, t \in \operatorname{Split}(p)$ of the same length, say n, there exists $u \in \operatorname{Split}(p)$ such that $s, t \in P_n^u$. Hence, for every $u \in \operatorname{Split}(q)$ there exists a uniquely determined $n_u \in A_u$ such that $\operatorname{Succ}_q(u) \subseteq P_{n_u}^u$.

We now define $S \in {}^\omega\omega$ such that for no $x \in [q]$ we have $x \geqslant^* S$. For every $u \in \operatorname{Split}(q)$ let

$$S(k_u - 1) = l_{n_u} + 1.$$

For $n \in \omega$ such that for no $u \in \operatorname{Split}(q)$ $n = k_u - 1$ let $S(n)$ be arbitrary. By the one-to-one choice of $\langle k_s : s \in {}^{<\omega}\omega \rangle$, S is well defined.

Suppose now $y \in [q]$. Choose $u, v \in \operatorname{Split}(q)$ such that $u \subset v \subset y$ and for no u' we have $u \subset u' \subset v$ and $u' \in \operatorname{Split}(q)$. Then $v \in P_{n_u}^u$ holds and hence by clause (3) in the definition of p we conclude:

$$y(k_u - 1) = v(k_u - 1) = l_{n_u} < S(k_u - 1).$$

By construction, $k_u - 1 > |u|$. But clearly there are infinitely many $u \in \operatorname{Split}(q)$ such that $u \subset y$. Hence, we have proved $\exists^\infty n(y(n) < S(n))$. $\square$

This completes the proof of Theorem 2.1. $\square$

3. Non-u-regular sets from rapid filters

Using ideas from [6] we will show that rapid filters can be used to construct non-u-regular sets.

Definition 3.1. A filter $\mathscr{F} \subseteq [\omega]^\omega$ is called *rapid* if and only if for every $x \in {}^\omega\omega$ there exists $a \in \mathscr{F}$ such that for every $n \in \omega$ we have

$$|a \cap (x(n) + 1)| < n.$$

Definition 3.2. For $\mathscr{F} \subseteq [\omega]^\omega$ let $\bar{\mathscr{F}} = \{x \in {}^\omega\omega : \exists a \in \mathscr{F}$ (x is the increasing enumeration of a)$\}$. $\mathscr{F}$ is called Σ^1_n if and only if $\bar{\mathscr{F}}$ is Σ^1_n.

Clearly, if $\mathscr{F}$ is a rapid filter then $\bar{\mathscr{F}}$ is dominating.

Theorem 3.3. *Suppose that there exists a rapid filter which is Σ^1_{n+1}. Then there exists a Π^1_n-set which is not u-regular.*

It is well known that in the constructible universe (or more generally in $L[r]$ for $r \in {}^\omega\omega$) a rapid filter exists which is Δ^1_2 (see [5]). Hence, we obtain the following corollary which is implicit in [7, 8].

Corollary 3.4. *Suppose $\exists r \in {}^\omega\omega(V = L[r])$. Then there exists a coanalytic set which is not u-regular.*

Proof of Theorem 3.3. Let $\mathscr{F} \subseteq [\omega]^\omega$ be a rapid filter which is Σ^1_{n+1}. There exists a Π^1_n-set $A \subseteq {}^\omega\omega \times {}^\omega\omega$ such that $\bar{\mathscr{F}} = \{y : \exists x(\langle x, y\rangle \in A)\}$. We want to transform A into a non-u-regular Π^1_n-subset of ${}^\omega\omega$.

To this end we use the function W from [6]. For $x \in {}^\omega\omega$ let $W(x) \in {}^\omega\omega$ be the following sequence:

$$\langle x(0) + 1 \text{ many 0's}, x(1) + 1 \text{ many 1's}, x(2) + 1 \text{ many 2's}, \ldots\rangle.$$

See [6, p. 109] for a formal definition of W. It is not difficult to see that W has the following properties:

(1) $\forall x \in {}^\omega\omega \forall n \in \omega(W(x)(n) \leq n)$.

(2) W is one to one, and for every $x \in {}^\omega\omega$ the formula expressing $\forall y(W(y) \neq x)$ and hence also that expressing $\exists y(W(y) = x)$ is arithmetic.

We now let

$$B = \{\langle x, y\rangle : \forall z(W(z) \neq x \vee \langle z, y\rangle \in A) \wedge \exists z(W(z) = x)\}.$$

By (2) above, B is a Π^1_n-set.

We next transform B by means of the function $V : {}^\omega\omega \times {}^\omega\omega \to {}^\omega\omega$, defined by the following clause:

$$V(x, y) = z \Leftrightarrow \forall n(z(2n) = y(n) \wedge z(2n + 1) = x(n) + y(n + 1)).$$

Clearly, V has the following properties:

(3) V is one to one.

(4) For every $z \in {}^\omega\omega$, the formula expressing $\exists \langle x, y\rangle(V(x, y) = z)$ is arithmetic.

We conclude that the following set is Π^1_n:

$$B_1 = \{z \in {}^\omega\omega : \forall \langle x, y\rangle(V(x, y) \neq z \vee \langle x, y\rangle \in B) \wedge \exists \langle x, y\rangle(V(x, y) = z)\}.$$

Claim. *B_1 is not u-regular.*

Proof of the Claim. (a) *B_1 is dominating*: Let $g \in {}^\omega\omega$ be arbitrary. We may assume that g is strictly increasing. Since $\mathscr{F}$ is rapid there exists $y \in \bar{\mathscr{F}}$ such that

$$\forall n(y(n) > g(2n)).$$

Choose $x \in {}^\omega\omega$ such that $\langle x, y\rangle \in A$. By construction,

$$z := V(\langle W(x), y\rangle) \in B_1$$

and $\forall n(z(2n) = y(n) > g(2n) \wedge z(2n+1) \geqslant y(n+1) > g(2n+1))$.

(b) *B_1 does not contain the branches of a uniform (even superperfect) tree*: For otherwise, let p be a counterexample. We will find $z_1, z_2 \in [p]$ such that

$$\{z_1(2n): n \in \omega\} \cap \{z_2(2n): n \in \omega\}$$

is finite. This will contradict our assumption that $\mathscr{F}$ is a filter. By induction we will construct $\langle s_i: i \in \omega\rangle$, $\langle t_i: i \in \omega\rangle$ such that

(5) $\forall i(s_i, t_i \in \mathrm{Split}(p) \wedge s_i \subset s_{i+1} \wedge t_i \subset t_{i+1} \wedge s_0 = t_0 = \mathrm{stem}(p))$.
(6) $\forall i(\{s_i(2n): \; 2n < |s_i|\} \cap \{t_i(2n): \; 2n < |t_i|\} \subseteq \{s_0(2n): \; 2n < |s_0|\} \cap \{t_0(2n): 2n < |t_0|\})$.

Letting $z_1 = \bigcup\{s_i: i \in \omega\}$, $z_2 = \bigcup\{t_i: i \in \omega\}$ we will be done.

Suppose that s_i, $t_i \in \mathrm{Split}(p)$ have been constructed.

If $|s_i|$ is even choose m so large that $s_i{}^\frown\langle m\rangle \in p$ and $m > \max\{t_i(2n: 2n < |t_i|\}$. Then let $s_{i+1} \in \mathrm{Split}(p)$ extending $s_i{}^\frown\langle m\rangle$ be arbitrary.

If $|s_i|$ is odd choose m so large that $s_i{}^\frown\langle m\rangle \in p$ and

$$m - |s_i| > \max\{t_i(2n): 2n < |t_i|\}.$$

Then let $s_{i+1} \in \mathrm{Split}(p)$ extending $\hat{s}_i\langle m\rangle$ be arbitrary. Note that by construction and property (1) of W then $s_{i+1}(|s_i| + 1) \geqslant m - |s_i|$ holds.

Now the construction of t_{i+1} is completely similar: Just replace s_i by t_i and t_i by s_{i+1} in the definition of s_{i+1}.

Then it is not difficult to see that (5) and (6) hold. □

This completes the proof of Theorem 3.3. □

Remark. The set B_1 might contain a perfect subset. Kechris [7, 8] has an example of a non-u-regular Π^1_1-set in L without a perfect subset.

4. Σ^1_2-K_σ-regularity implies Σ^1_2-u-regularity

The following characterization of Σ^1_2-K_σ-regularity is implicit in [7, 8], and explicit in [5].

Theorem 4.1. *Σ^1_2-K_σ-regularity holds if and only if for every $r \in {}^\omega\omega$, ${}^\omega\omega \cap L[r]$ is $\leqslant^*$-bounded.*

In order to prove the direction "$\Rightarrow$" (by contraposition) the non-u-regular set $\mathscr{C}^r_1$ in $L[r]$ from [7] or the set $B^r_1 \in L[r]$ from Section 3 of this paper which both are Π^1_1 in V can be used. If for some $r \in {}^\omega\omega$, ${}^\omega\omega \cap L[r]$ is unbounded in V, then $\mathscr{C}^r_1$ or B^r_1 are unbounded in V.

The direction "$\Leftarrow$" is proved in [8] using a modification of the game argument from the proof that analytic sets are K_σ-regular.

An easy modification of the proof of the next theorem gives an alternative proof of this direction.

In [5] it was shown that Σ^1_2-K_σ-regularity follows from either Σ^1_2-categoricity and hence Σ^1_2-measurability (by [1] or [12]) or Σ^1_2-Ramsey-property. What is the relation between Σ^1_2-u-regularity and these?

Theorem 4.2. *Σ^1_2-K_σ-regularity implies Σ^1_2-u-regularity.*

Proof. Suppose Σ^1_2-K_σ-regularity holds. Let A be a dominating Σ^1_2-set. The proof that A contains the branches of a uniform tree is divided into two cases:

Case 1: $\exists r \in {}^\omega\omega(\omega_1^{L[r]} = \omega_1)$.

Choose $r \in {}^\omega\omega$ witnessing Case 1 such that the real coding the definition of A belongs to $L[r]$. It is well known (see [4, pp. 520, 526]) that every Σ^1_2-set is the union of $\aleph_1$ Borel sets. Moreover, this decomposition is absolute for every ZFC-model computing ω_1 correct. This follows from its construction using the Shoenfield tree. So by Case 1, if

$$A = \bigcup \{A_\alpha \colon \alpha < \omega_1\}$$

is the decomposition of A into Borel sets in $L[r]$, then this is the true decomposition.

Now if for some $\alpha < \omega_1$ A_α is dominating in $L[r]$, then by Theorem 1.4 applied in $L[r]$ there exists a uniform tree $p \in L[r]$ such that $[p] \subseteq A_\alpha$. However, the formula expressing this inclusion is Π^1_1 and hence by Shoenfield absoluteness it holds in V, and we are done.

On the other hand, if for no $\alpha < \omega_1 A_\alpha$ is dominating in $L[r]$ there exists $\langle f_\alpha \colon \alpha < \omega_1 \rangle \in L[r]$ such that f_α witnesses this for A_α, i.e.

$$\forall x \in A_\alpha (x \not\geqslant^* f_\alpha).$$

Again this formula is Π^1_1 and hence it holds in V. But by assumption that Σ^1_2-K_σ-regularity holds and Theorem 4.1, in V there exists $f \in {}^\omega\omega$ such that $\forall\alpha < \omega_1 (f \geqslant^* f_\alpha)$. But then clearly f witnesses that A is not dominating in V, a contradiction.

(An analogous argument proves Σ^1_2-K_σ-regularity from "$\forall r \in {}^\omega\omega({}^\omega\omega \cap L[r]$ is $\leqslant^*$-bounded)" in Case 1.)

Case 2: $\forall r \in {}^{\omega}\omega(\omega_1^{L[r]} < \omega_1)$.

We will show that in this case Σ_2^1-u-regularity always holds (the same proof shows that this is true for Σ_2^1-K_σ-regularity). Clearly the formula expressing "A is dominating" is Π_3^1. By Shoenfield absoluteness Π_3^1-formulas are downward absolute, hence A is dominating in every submodel of V containing all ordinals and the code for the definition of A. Let $r \in {}^{\omega}\omega$ be this code.

In $L[r]$, A is the union of $\aleph_1$ Borel sets, say

$$A = \bigcup \{A_\alpha : \alpha < \omega_1^{L[r]}\}. \tag{1}$$

As now $\omega_1^{L[r]}$ is countable, in general this decomposition is not absolute. However, it is true that it is an initial segment of the true decomposition (in V), i.e. for every $\alpha < \omega_1^{L[r]} A_\alpha \subseteq A$ holds in V. This follows from the construction of the decomposition.

Now comes the main trick: In $L[r]$, let P be a finite (or countable) support iteration of length $\omega_2^{L[r]}$ of Hechler forcing (or of any definable proper forcing of size continuum adding a dominating real – if the forcing is not ccc then P has to be a countable support iteration). Remember that Hechler conditions are pairs $\langle s, f\rangle \in {}^{<\omega}\omega \times {}^{\omega}\omega$ such that $s \subseteq f$ and $\langle s, f\rangle$ extends $\langle t, g\rangle$ if and only if $s \supseteq t$ and $\forall n \geqslant |t|(f(n) \geqslant g(n))$. By [2] we may assume that

$$L[r] \models |P| = \omega_2.$$

But by Case 2, ω_1 is inaccessible in $L[r]$, and hence $(2^{\omega_2})^{L[r]} < \omega_1$. Hence, in V, $P^{L[r]}$ has countably many dense sets in $L[r]$, and so we may choose $\langle r_\delta : \delta < \omega_2^{L[r]}\rangle \in V$ which is $P^{L[r]}$-generic over $L[r]$. As forcing with P preserves ω_1 (1) is the decomposition into $\aleph_1$ Borel sets in $L[r][\langle r_\delta : \delta < \omega_2^{L[r]}\rangle]$. By the remark about downward absoluteness of Π_3^1-formulas we know that A is dominating in $L[r][\langle r_\delta : \delta < \omega_2^{L[r]}\rangle]$.

We now can repeat the argument from Case 1: If for some $\alpha < \omega_1^{L[r]}$ B_α is dominating in $L[r][\langle r_\delta : \delta < \omega_2^{L[r]}\rangle]$, by Theorem 1.4 and the Shoenfield absoluteness we are done. Otherwise, for every $\alpha < \omega_1^{L[r]}$ we had $f_\alpha \in {}^{\omega}\omega$ witnessing that B_α is not dominating. But as for every f_α there exists r_δ such that $r_\delta \geqslant^* f_\alpha$ and $\forall \delta < \nu < \omega_2^{L[r]} (r_\delta <^* r_\nu)$ we conclude that there exists $\delta < \omega_2^{L[r]}$ such that

$$\forall \alpha < \omega_1^{L[r]} (r_\delta \geqslant^* f_\alpha).$$

But then r_δ witnesses that A is not dominating in $L[r][\langle r_\delta : \delta < \omega_2^{L[r]}\rangle]$, a contradiction.

Remark. Howard Becker pointed out to me that a slight modification of the argument above shows that every dominating analytic set contains a dominating Borel set: Let $\langle N, \in\rangle$ be a countable transitive model elementarily embeddable into $\langle V_\kappa, \in\rangle$ for some large enough cardinal κ, such that N contains the code for A. Now let P be a finite support iteration of length ω_2 of Hechler forcing in the sense of N. In V choose $\langle r_\delta : \delta < \omega_2^N\rangle$ P-generic over N. Now the formula saying "A is dominating" is Π_2^1, it holds in N by elementarity and in $N[\langle r_\delta : \delta < \omega_2^N\rangle]$ by Shoenfield's Lemma in N.

Hence, one B_α from A's decomposition into $\aleph_1$ Borel sets is dominating in $N[\langle r_\delta: \delta < \omega_2^N \rangle]$, hence in N by Shoenfield absoluteness, and hence in V by elementarity.

Question 1. Is Σ^1_2-u-regularity equivalent to Σ^1_2-K_σ-regularity?

Question 2. Does every dominating analytic set contain a dominating closed set?

By Theorems 4.1 and 4.2 clearly Σ^1_2-K_σ- and hence Σ^1_2-u-regularity hold under Martin's axiom and $\neg$CH. Note that this contrasts with the situation for the perfect set property: Perfect set property for all Π^1_1-sets implies that ω_1 is inaccessible in L (see [4]). From [11] and Theorem 3.3 it follows that Π^1_2-u-regularity together with Σ^1_2-measurability implies that ω_1 is inaccessible in L. The proof of this is analogous to the one for K_σ-regularity in [6]. However, the following question is open.

Question 3. Does Π^1_2-K_σ- or Π^1_2-u-regularity imply that ω_1 is inaccessible in L?

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